

Aspherical neighborhoods on arithmetic surfaces: the local case

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Abstract

On arithmetic surfaces over local rings of integers we examine whether any geometric point has a basis of étale neighborhoods whose \mathfrak{c} -completed étale homotopy types are of type $K(\pi, 1)$ for a given full class \mathfrak{c} of finite groups.

1 Introduction

We fix a base scheme B which is the spectrum of an excellent Dedekind ring of dimension one. By an arithmetic surface over B we mean an irreducible, normal scheme X/B of dimension 2 which is flat and of finite type with geometrically connected generic fibre. We say that a connected, locally noetherian scheme U is $K(\pi, 1)$ with respect a full class of finite groups \mathfrak{c} if the \mathfrak{c} -completion of the étale homotopy type of U has trivial π_n for $n \geq 2$.

Theorem 1. *Let B be the spectrum of the ring of integers of a local field and $X \rightarrow B$ an arithmetic surface. Assume that the order of every group in \mathfrak{c} is prime to the residue characteristic of B and that for all but finitely many primes ℓ with $\mathbb{Z}/\ell\mathbb{Z} \in \mathfrak{c}$ the extension $B[\mu_\ell] \rightarrow B$ is a \mathfrak{c} -extension. Then every geometric point \bar{x} of X has a basis of étale neighborhoods which are $K(\pi, 1)$ with respect to \mathfrak{c} .*

Unlike in the case of varieties treated by Friedlander in [3] it is not possible to work with good Artin neighborhoods in the arithmetic situation. In fact in [7] it is shown that on arithmetic schemes étale bases of neighborhoods which are (the arithmetic analogues of) good Artin neighborhoods never exist.

In view of future work on global arithmetic surfaces we formulate most results for arithmetic surfaces over an arbitrary Dedekind scheme B . It is only in Sections 9 and 11 that we restrict our attention to the local case.

2 Tidy divisors on arithmetic surfaces

A divisor D on an arithmetic surface X/B is *tidy* at a point $x \in D$ if it has normal crossings at x and intersects each vertical divisor of X passing through x transversally. A *tidy* divisor on X is a divisor which is tidy at every point of X . In particular, the horizontal irreducible components of a tidy divisor do not intersect. For a proper closed subscheme $Z \subseteq X$ we say that a closed point $z \in Z$

is a *special point* of Z if either Z is not a tidy divisor at z or Z is tidy at z and Z is singular at z . The special points of a tidy divisor D are precisely the points where two irreducible components of D intersect. If D is not tidy but snc, the special points are the singular points of D_{red} and the points where D intersects a vertical divisor non-transversally.

Let $Z \subseteq X$ be a proper closed subscheme of an arithmetic surface X/B . We define a *minimal desingularization* of (X, Z) to be a morphism of pairs $\phi : (X', Z') \rightarrow (X, Z)$ such that X' is regular at all points of Z , $(X' - Z') \rightarrow (X - Z)$ is an isomorphism and ϕ is universal with this property. Minimal desingularizations of (X, Z) exist by [9] and are unique up to unique isomorphism.

Definition 2. Let X/B be an arithmetic surface and $Z \subseteq X$ a proper closed subscheme. A tidy desingularization $(X', Z') \rightarrow (X, Z)$ of (X, Z) is a birational morphism $X' \rightarrow X$ such that Z' is a tidy divisor of X' and $(X', Z') \rightarrow (X, Z)$ factors as

$$(X', Z') = (X_0, Z_0) \rightarrow (X_1, Z_1) \rightarrow \dots \rightarrow (X_n, Z_n) \rightarrow (X, Z),$$

where $X_n \rightarrow X$ is the minimal desingularization of (X, Z) and for $i = 1, \dots, n$ the morphisms $(X_{i-1}, Z_{i-1}) \rightarrow (X_i, Z_i)$ are blowups of X_i in special points of Z_i .

Proposition 3. Let X/B be an arithmetic surface and $Z \subset X$ a proper closed subscheme. Then a tidy desingularization of (X, Z) exists.

Proof. We may assume that X is regular at all points of $X - Z$ as X is singular in at most a finite set of closed points, which we can remove from X if they do not lie on Z . By [1], Theorems 0.1 and 0.2 there is a desingularization $(X', Z') \rightarrow (X, Z)$ which is an isomorphism over the complement of Z such that Z' is an snc-divisor. Moreover, we can assume that $(X', Z') \rightarrow (X, Z)$ is obtained from the minimal desingularization by successive blow-ups in singular, hence special, points. Let D' be the union of Z' with the finitely many vertical prime divisors containing the points where Z' intersects a vertical divisor non-transversally. After removing from X' all points of D' which are not contained in Z' and where D' is singular, we may assume that the special points of D' are contained in Z' . By construction, they coincide with the singular points of D' . Blowing up in singular points of D' , we achieve that D' is an snc-divisor. This is equivalent to saying that Z' is tidy. \square

For an arithmetic surface X/B and a tidy divisor D on X we are interested in \mathfrak{c} -coverings of (X, D) for a full class of finite groups whose orders are invertible on X . Here, a \mathfrak{c} -covering of (X, D) is a finite morphism of pairs $(X_1, D_1) \rightarrow (X, D)$ whose restriction to the complement of D_1 is an étale \mathfrak{c} -covering of $X - D$. Note that any \mathfrak{c} -covering is tame. For a tame covering $(X_1, D_1) \rightarrow (X, D)$, D_1 is not a tidy divisor of X_1 in general. But we find a tidy desingularization $(X', D') \rightarrow (X_1, D_1)$ using proposition 3.

Definition 4. A desingularized \mathfrak{c} -covering $(X', D') \rightarrow (X, D)$ is the composition of a \mathfrak{c} -covering $(X_1, D_1) \rightarrow (X, D)$ and a tidy desingularization $(X', D') \rightarrow (X_1, D_1)$.

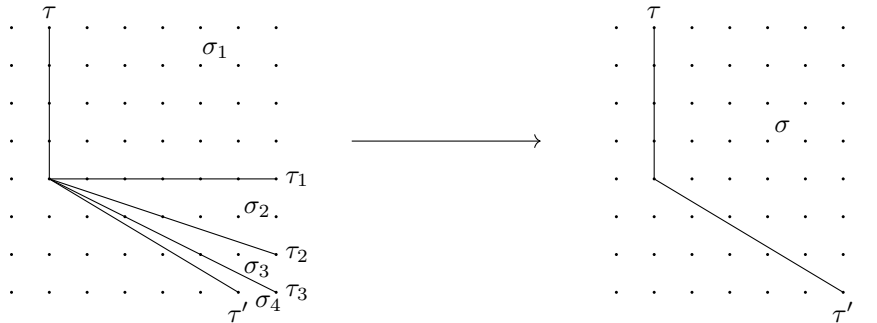
If \mathfrak{c} is not specified, we tacitly assume it to be the class of all finite groups of order invertible on X and speak of desingularized tame coverings instead of desingularized \mathfrak{c} -coverings. Furthermore, we call the exceptional divisor of $X' \rightarrow X_1$ the generalized exceptional divisor of $(X', D') \rightarrow (X, D)$.

3 Exceptional fibres

Let us call curve a noetherian scheme whose irreducible components are one-dimensional. We say that a curve C is a *chain of \mathbb{P}^1 's* if its irreducible components C_1, \dots, C_n are isomorphic to \mathbb{P}_k^1 for some field k , for $i = 1, \dots, n-1$ the curve C_i intersects C_{i+1} in exactly one point, which is moreover k -rational, and $C_i \cap C_j$ is empty for $|i - j| \geq 2$. If C is a closed subscheme of another curve C_0 , we say that C is a *bridge of \mathbb{P}^1 's* in C_0 if C is a chain of \mathbb{P}^1 's and C intersects exactly two of the remaining irreducible components of C_0 and this intersection takes place in two k -rational points $c_1 \in C_1$ and $c_n \in C_n$.

Proposition 5. *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $(X_1, D_1) \rightarrow (X, D)$ be a tame covering of (X, D) and $(X'_{\min}, D'_{\min}) \rightarrow (X_1, D_1)$ the minimal desingularization of (X_1, D_1) . Then D'_{\min} is a tidy divisor and the exceptional fibres of $X'_{\min} \rightarrow X_1$ are bridges of \mathbb{P}^1 's in D'_{\min} . In particular, $(X'_{\min}, D'_{\min}) \rightarrow (X, D)$ is a tidy desingularization of (X, D) . Moreover, for any other desingularized tame covering $(X', D') \rightarrow (X, D)$ the generalized exceptional fibres are bridges of \mathbb{P}^1 's in D' , as well.*

Proof. We view (X, D) as a log scheme with log structure $m_D \rightarrow \mathcal{O}_X$ given by the divisor D . Since D has normal crossings, (X, D) is log-regular and the corresponding log structure is toric. The tame covering $(X_1, D_1) \rightarrow (X, D)$ is Kummer étale. In particular, it is log-smooth and thus (X_1, D_1) is log-regular and the corresponding log structure toric, as well. In section 10 of [8] Kato associates a fan F_D to the log scheme (X, D) . In this context a fan is a monoidal space which is locally isomorphic to $\text{Spec } P$ for a sharp monoid P . As a topological space the fan F_D is the subspace $\{x \in X \mid I(x, m_D) = \mathfrak{m}_x\}$ of X , where $I(x, m_D)$ is the ideal generated by $m_{D,x} \setminus m_{D,x}^\times$. The structure sheaf is given by the inverse image of $m_D \setminus \mathcal{O}_X^\times$. Since the log structure of (X_1, D_1) is toric, the fan F_D corresponds to a classical fan Δ , i.e. a fan of convex polyhedral cones in a two-dimensional lattice L as described in [5]. We may work locally and thus assume that Δ consists of one two-dimensional cone σ together with its two one-dimensional faces τ and τ' and $\{0\}$. The faces τ and τ' correspond to prime divisors P and P' on X_1 constituting the irreducible components of D_1 (see [8], Corollary 11.8). They intersect in one point $x_1 \in X_1$, which is the only possibly singular point of X_1 . By [5], section 2.6 we can find a subdivision Δ' of Δ in cones which are isomorphic to \mathbb{N}^2 . In dimension 2 a subdivision of σ is given by inserting additional rays $\tau_1, \dots, \tau_{n-1}$ forming the faces of cones $\sigma_1, \dots, \sigma_n$.



By [8], 10.4. this provides us with a resolution of singularities $(X', D') \rightarrow (X_1, D_1)$ such that D' has strictly normal crossings. The exceptional fibre consists of prime divisors E_1, \dots, E_{n-1} corresponding to the rays $\tau_1, \dots, \tau_{n-1}$ and E_i intersects E_{i+1} in one point corresponding to the cone σ_i . The strict transforms of P and P' correspond to the rays τ and τ' . Hence, P intersects τ_1 in one point and P' intersects τ_{n-1} in one point. It remains to see that the E_i are rational. By the proof of [8], Proposition 9.9 we have

$$X' = X_1 \times_{\mathbb{Z}[\Delta]} \text{Spec } \mathbb{Z}[\Delta'].$$

The exceptional fibre is thus given by

$$\text{Spec } k(x_1) \times_{\mathbb{Z}[\Delta]} \text{Spec } \mathbb{Z}[\Delta'].$$

Locally this is the spectrum of $k(x_1)[\sigma_i^\vee/\sigma^\vee]$, which is readily checked to be rational. \square

Corollary 6. *In the situation of Proposition 5 let x_1 be a special point of D_1 . Let Z_1 be an irreducible component of D_1 containing x_1 . Denote by Z' its strict transfer in X'_{\min} and by Z its image in X . Let E_1, \dots, E_n be the irreducible components of the exceptional fibre of $X'_{\min} \rightarrow X_1$ above x_1 such that E_i intersects E_{i+1} and Z' intersects E_1 . Then above an open neighborhood of x_1 the pullback of Z to X'_{\min} is given by*

$$a_0 Z' + a_1 E_1 + \dots + a_n E_n$$

with $a_0 > a_1 > \dots > a_n > 0$.

Proof. Denote by b the image of x in B and by π the morphism $X'_{\min} \rightarrow X$. In order to simplify notation, we set $E_0 := Z'$. By the projection formula and Proposition 5 we have

$$0 = \pi^* Z \cdot E_n = (a_0 E_0 + a_1 E_1 + \dots + a_n E_n) \cdot E_n = [k(x_1) : k(b)](a_{n-1} + a_n E_n^2).$$

Since the desingularization $X'_{\min} \rightarrow X_1$ is minimal, E_n cannot be a -1 -curve and thus $E_n^2 < -1$. (The self-intersection of E_n has to be negative by [10], chapter 9, Theorem 1.27.) Hence,

$$a_{n-1} = -a_n E_n^2 > a_n.$$

By induction we may assume that $a_{i+1} < a_i$ for $0 < k \leq i < n$. Again by the projection formula we obtain

$$0 = \pi^* Z \cdot E_k = [k(x_1) : k(b)](a_{k-1} + a_k E_k^2 + a_{k+1}).$$

By induction and using $E_k^2 \leq -2$ we conclude that

$$a_{k-1} = -a_{k+1} - a_k E_k^2 > -a_{k+1} + 2a_k > a_k. \quad \square$$

Corollary 7. *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $(X_1, D_1) \rightarrow (X, D)$ be a tame covering of (X, D) and $(X', D') \rightarrow (X_1, D_1)$ a desingularization of (X_1, D_1) . Assume that every irreducible component of an exceptional fibre of $(X', D') \rightarrow (X_1, D_1)$ intersects the other irreducible components of D' in at least two points. Then $(X', D') \rightarrow (X_1, D_1)$ is a tidy desingularization.*

Proof. We can factor $(X', D') \rightarrow (X_1, D_1)$ as

$$(X', D') := (X'_0, D'_0) \rightarrow (X'_1, D'_1) \rightarrow \dots \rightarrow (X'_n, D'_n) \rightarrow (X_1, D_1),$$

where $(X'_n, D'_n) \rightarrow (X_1, D_1)$ is the minimal desingularization of (X_1, D_1) and for $i = 1, \dots, n$ the morphism $(X'_{i-1}, D'_{i-1}) \rightarrow (X'_i, D'_i)$ is the blowup of X'_i in a closed point d'_i of D'_i . By proposition 5 the minimal desingularization $(X'_n, D'_n) \rightarrow (X_1, D_1)$ is a tidy desingularization. Moreover, blowing up in closed points does not destroy the tidiness of a divisor. Hence, D'_i is a tidy divisor of X'_i for all $i = 0, \dots, n$. Suppose that $(X', D') \rightarrow (X_1, D_1)$ is not a tidy desingularization. Then there is an index i such that d'_i is not a special point of D'_i , i.e., d'_i is a regular point of D'_i . Let i_0 be the smallest such index. Then the exceptional fibre of $(X'_{i_0-1}, D'_{i_0-1}) \rightarrow (X'_{i_0}, D'_{i_0})$ has only one intersection point with the other irreducible components of D'_{i_0-1} . This does not change by blowing up D'_{i_0-1} in special points. We thus obtain a contradiction. \square

Let D be a tidy divisor on an arithmetic surface X and $\bar{x} \rightarrow U = X - D$ a geometric point. We denote by $\mathfrak{J}_{X,D,\bar{x}}$ the category of all desingularized \mathfrak{c} -coverings $(X', D') \rightarrow (X, D)$ together with a geometric point $\bar{x} \rightarrow X'$ such that $\bar{x} \rightarrow X' \rightarrow X$ coincides with the fixed geometric point $\bar{x} \rightarrow X$. The explicit description of the exceptional fibres in Lemma 5 enables us to prove that the category $\mathfrak{J}_{X,D,\bar{x}}$ is cofiltered:

Lemma 8. *The following assertions hold:*

- (i) *If $(X', D') \rightarrow (X, D)$ and $(X'', D'') \rightarrow (X', D')$ are both desingularized \mathfrak{c} -coverings, the composite $(X'', D'') \rightarrow (X, D)$ is again a desingularized \mathfrak{c} -covering.*
- (ii) *If $(X', D') \rightarrow (X, D)$ and $(X'', D'') \rightarrow (X, D)$ are desingularized \mathfrak{c} -coverings, there is a commutative diagram of desingularized \mathfrak{c} -coverings*

$$\begin{array}{ccccc} & & (X', D') & & \\ & \nearrow & & \searrow & \\ (X''', D''') & & & & (X, D) \\ & \searrow & & \nearrow & \\ & & (X'', D'') & & \end{array}$$

Proof. (i). Let X_1 be the normalization of X in $K(X')$ and X_2 its normalization in $K(X'')$. Furthermore, denote by X'_1 the normalization of X' in $K(X'')$. We obtain a cartesian diagram

$$\begin{array}{ccccc} D'' & \longrightarrow & D_2 & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & X_2 & \longrightarrow & X. \end{array}$$

Since $U' = X' - D'$ is the normalization of $U = X - D$ in $K(X')$ and $U'' = X'' - D''$ is the normalization of U' in $K(X'')$, U'' is also the normalization of U in $K(X'')$. It is thus an open subscheme of X_2 and $U'' \rightarrow U$ is a finite étale \mathfrak{c} -covering as finite étale \mathfrak{c} -coverings are stable under composition. Hence, $X'' \rightarrow$

X_2 is birational and an isomorphism on U'' . Moreover, $D'' \subseteq X''$ is a tidy divisor. The only remaining question is whether $X'' \rightarrow X_2$ is obtained from the minimal desingularization of (X_2, D_2) by successively blowing up in special points. By Corollary 7 it suffices to show that every irreducible component of an exceptional fibre of $X'' \rightarrow X_2$ meets the other irreducible components of D'' in at least two points. The morphisms $X'' \rightarrow X'$ and $X' \rightarrow X$ factor as

$$\begin{aligned} (X'', D'') &= (Y_0, Z_0) \rightarrow \dots \rightarrow (Y_n, Z_n) = (X'_1, D'_1) \rightarrow (X', D'), \\ (X', D') &= (Y_{n+1}, Z_{n+1}) \rightarrow \dots \rightarrow (Y_m, Z_m) = (X_1, D_1) \rightarrow (X, D), \end{aligned}$$

where $(Y_n, Z_n) \rightarrow (X'_1, D'_1)$ and $(Y_m, Z_m) \rightarrow (X_1, D_1)$ represent the minimal desingularizations of (X'_1, D'_1) and (X_1, D_1) , respectively, and for $i = 1, \dots, n$ and $i = n+2, \dots, m$ the morphism $(Y_{i-1}, Z_{i-1}) \rightarrow (Y_i, Z_i)$ is the blowup of Y_i in a special point z_i of Z_i . Let E be an irreducible component of an exceptional fibre of $X'' \rightarrow X_2$. There is $i \in \{1, \dots, n\} \cup \{n+2, \dots, m\}$ such that the image of E in Y_{i-1} is one-dimensional and its image in Y_i is a closed point. This closed point is precisely the point z_i and we obtain a finite morphism from E to the exceptional fibre of $Y_{i-1} \rightarrow Y_i$ in X'' . Since $X_{i-1} \rightarrow X_i$ is the blowup of X_i in z_i and z_i is a special point, its exceptional fibre intersects the other irreducible components of Z_{i-1} in two points. The intersection points of E contain the preimages of these two points and thus there are at least two intersection points.

(ii). Let K''' be the compositum of $K(X')$ and $K(X'')$ and X_3 the normalization of X in K''' . This defines a \mathfrak{c} -covering $(X_3, D_3) \rightarrow (X, D)$. We obtain rational maps $X_3 \dashrightarrow X'$ and $X_3 \dashrightarrow X''$, which, restricted to $U_3 = X_3 - D_3$, are finite étale \mathfrak{c} -coverings of $U' = X' - D'$ and $U'' = X'' - D''$, respectively. Using elimination of indeterminacies and the existence of tidy desingularizations we find a desingularization $(X''', D''') \rightarrow (X_3, D_3)$ dominating (X', D') and (X'', D'') such that D''' is tidy. Suppose there is an irreducible component E of an exceptional fibre of X''' with only one intersection point with the other irreducible components of D''' . Let us write

$$(X''', D''') = (X_0''', D_0''') \rightarrow \dots \rightarrow (X_n''', D_n''') \rightarrow (X_3, D_3),$$

where $(X_n''', D_n''') \rightarrow (X_3, D_3)$ is the minimal desingularization of (X_3, D_3) and for $i = 1, \dots, n$ the morphism $X_{i-1}''' \rightarrow X_i'''$ is the blowup of X_i''' in a closed point $d_i \in D_i'''$. There is $i \in \{1, \dots, n\}$ such that the image of E is the point d_i and the image of E in X_{i-1}''' is the exceptional fibre E_i of $X_{i-1}''' \rightarrow X_i'''$. Since E has only one intersection point, the same holds for E_i . Furthermore, the blowup points d_k for $k = 1, \dots, i-1$ must not lie above E_i except possibly above the intersection point z_i of E_i with the other irreducible components. One checks that after blowing up in z_i the strict transform of E_i is still a -1 -curve. Therefore, we can contract E . Moreover, by similar arguments as in the proof of part (i) the image of E in X' as well as in X'' is a point. Hence, the contraction still factors through $X' \rightarrow X$ and $X'' \rightarrow X$. After finitely many contractions we may assume that all irreducible components of exceptional fibres of $X''' \rightarrow X_3$ have at least two intersection points. Then the same holds for the generalized exceptional fibres of $X''' \rightarrow X'$ and of $X''' \rightarrow X''$ as these are contained in the exceptional fibres of $X''' \rightarrow X_3$. The assertion now follows from Corollary 7. \square

4 Arithmetic surfaces with enough tame coverings

Let X/B be an arithmetic surface and $D \subset X$ a tidy divisor. Fix a full class \mathfrak{c} of finite groups whose orders are invertible on X . The following property is crucial for showing that $U = X - D$ is $K(\pi, 1)$ with respect to \mathfrak{c} :

Definition 9. We say that (X, D) has enough tame coverings at a closed point x of D if for every irreducible component C of D passing through x there is $f \in K(X)^\times$ with support in D such that $\deg_C(f) > 0$ and $\deg_W(f) = 0$ for any other prime divisor W passing through x . We say that (X, D) has enough tame coverings if it has enough tame coverings at every closed point of D .

If (X, D) has enough tame coverings at a point x and C is an irreducible component of D passing through x , we can construct \mathfrak{c} -coverings of (X, D) of arbitrarily high ramification index in C by taking the normalization of X in a function field extension $K(X)(\sqrt[n]{f})|K(X)$ with f chosen as in Definition 9. In a neighborhood of x this covering ramifies only in C .

Lemma 10. Let $\pi : (X', D') \rightarrow (X_1, D_1) \rightarrow (X, D)$ be a desingularized \mathfrak{c} -covering. If (X, D) has enough tame coverings, the same holds for (X', D') .

For the proof of Lemma 10 we need to investigate the multiplicities of the irreducible components of the pullback to X' of a prime divisor on X .

Definition 11. Let $f : (X', D') \rightarrow (X, D)$ be a desingularized \mathfrak{c} -covering. Let $x' \in D'$ be a closed point and denote by $x \in D$ the image of x' in X . Let us call D_1, \dots, D_n (necessarily $n = 1$ or $n = 2$) the irreducible components of D passing through x and D'_1, \dots, D'_m ($m \leq n$) the irreducible components of D' passing through x' . Restricting f to a suitable neighborhood of x' , the pullback of Cartier divisors via f induces a homomorphism

$$\mathbb{Q} \cdot D_1 \oplus \dots \oplus \mathbb{Q} \cdot D_n \rightarrow \mathbb{Q} \cdot D'_1 \oplus \dots \oplus \mathbb{Q} \cdot D'_m.$$

We call this morphism multiplicity homomorphism at x' and its transformation matrix with respect to the above bases multiplicity matrix at x' .

Multiplicity homomorphisms are compatible with composition. If $(X'', D'') \rightarrow (X', D')$ is another morphism as above and x'' a closed point of D'' mapping to $x' \in D'$, the multiplicity homomorphism of $(X'', D'') \rightarrow (X', D')$ at x'' is the composition of the multiplicity homomorphism of $(X'', D'') \rightarrow (X, D)$ at x'' and the multiplicity homomorphism of $(X', D') \rightarrow (X, D)$ at x' .

Lemma 12. Let $(X', D') \rightarrow (X, D)$ be the blowup of X in a special point x of D . Then all multiplicity homomorphisms are surjective.

Proof. Denote by D_1 and D_2 the irreducible components of D passing through x and by D'_1 and D'_2 their strict transforms in X' . Furthermore, let E denote the singular fibre of $X' \rightarrow X$. On $E \subseteq D'$ there are two points x'_1 and x'_2 where D' is singular, namely the respective intersection points with D'_1 and D'_2 . The pullback of D_i is given by $D'_i + E$. Hence, the intersection matrix at x'_1 as well as at x'_2 (with respect to the bases $\{(D_1, D_2), (D'_1, E)\}$ and $\{(D_1, D_2), (E, D'_2)\}$, respectively) is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which is invertible. If $x' \in E$ is a nonsingular point of D' , its multiplicity matrix is

$$\begin{pmatrix} 1 & 1 \end{pmatrix},$$

which is nonzero and thus its multiplicity homomorphism is surjective. The multiplicity homomorphism at any other closed point of D' is the identity. \square

Lemma 13. *Let $\pi : (X', D') \rightarrow (X_1, D_1) \rightarrow (X, D)$ be a desingularized \mathfrak{c} -covering. Then all multiplicity homomorphisms are surjective.*

Proof. By Lemma 12 we may assume that $X' \rightarrow X_1$ is the minimal desingularization of X_1 . Let $x' \in D'$ be a closed point and denote by x_1 and x the image of x' in X_1 and X , respectively. If x' is a regular point of D' , there is only one irreducible component of D' passing through x' . Hence, the multiplicity homomorphism at x' is surjective if and only if it is nonzero, which is clear by taking the pullback of any irreducible component of D passing through x .

Suppose that x' is a singular point of D' . Then also x_1 and x are singular points of D_1 and D , respectively. There are two irreducible components Z_1 and W_1 of D_1 passing through x_1 mapping to the irreducible components W and Z of D passing through x . According to Corollary 6 we have in a neighborhood of x'

$$\pi^*Z = a_0Z' + a_1E_1 + \dots + a_nE_n$$

with $a_0 > a_1 > \dots > a_n > 0$ and

$$\pi^*W = b_1E_1 + \dots + b_nE_n + b_{n+1}W'$$

with $b_1 < \dots < b_n < b_{n+1}$ and where W' denotes the strict transform of W_1 in X' . Setting $E_0 := Z'$ and $E_{n+1} := W'$ we know that there is an integer i with $0 \leq i \leq n$ such that x' is the intersection point of E_i with E_{i+1} . The multiplicity matrix at x' is

$$\begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix}$$

and

$$\det \begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix} = a_i b_{i+1} - a_{i+1} b_i > a_i b_i - a_i b_i = 0$$

as $a_{i+1} < a_i$ and $b_{i+1} > b_i$. Therefore, also in this case the multiplicity homomorphism is surjective. \square

Proof of Lemma 10. Assume that (X, D) has enough tame coverings. Let $x' \in D'$ be a closed point and Z' an irreducible component of D' passing through x' . We have to find $f' \in K(X')^\times$ with support in D' such that $\deg_{Z'}(f') > 0$ and $\deg_{C'}(f') = 0$ for all other irreducible components C' of D' passing through x' . Let Z_1, \dots, Z_r (for $r = 1$ or $r = 2$) denote the irreducible components of D passing through the image point $x \in D$ of x' . Since (X, D) has enough tame coverings, for $i = 1, \dots, r$ there is $f_i \in K(X)^\times$ with support in D such that $\deg_{Z_i}(f_i) > 0$ and $\deg_{Z_j}(f_i) = 0$ for $i \neq j$. The projections of $\text{div } f_i$ to

$$\mathbb{Q} \cdot Z_1 \oplus \dots \oplus \mathbb{Q} \cdot Z_r$$

constitute a basis of this vector space. Let $Z' = Z'_1, \dots, Z'_s$ denote the irreducible components of D' passing through x' . Lemma 13 provides the surjectivity of the multiplicity homomorphism

$$\phi_{x'} : \mathbb{Q} \cdot Z_1 \oplus \dots \oplus \mathbb{Q} \cdot Z_r \rightarrow \mathbb{Q} \cdot Z'_1 \oplus \dots \oplus \mathbb{Q} \cdot Z'_s$$

at x' induced by pullback. We obtain integers d, k_1, \dots, k_r with $d > 0$ such that

$$d \cdot Z'_1 = \phi_{x'}(k_1 \operatorname{div} f_1 + \dots k_r \operatorname{div} f_r).$$

In other words, setting $f = f_1^{k_1} \cdot \dots \cdot f_r^{k_r}$ we have in a neighborhood of x'

$$\operatorname{div} f = d \cdot Z',$$

what we wanted to prove. \square

5 Absolute cohomological purity

Let (X, Z) be a regular pair of codimension c and m a positive integer invertible on X . Set $\Lambda = \mathbb{Z}/m\mathbb{Z}$. The absolute cohomological purity theorem proved by Gabber in [4] provides a canonical isomorphism

$$\underline{H}_Z^n(\Lambda) \cong \begin{cases} 0 & \text{for } n \neq 2c \\ \Lambda_Z(-c) & \text{for } n = 2c. \end{cases}$$

which is induced from the cycle class map which maps $1 \in \Lambda$ to the fundamental class $s_{Z/X} \in H_Z^{2c}(X, \Lambda(c))$. Since the étale site of a scheme is equivalent to the étale site of its reduction, the statement also holds if only X_{red} and Z_{red} are regular. We call (X, Z) a regular pair if $(X_{\text{red}}, Z_{\text{red}})$ is a regular pair in the usual sense. Taking into account that the pullback of the fundamental class $s_{Z_{\text{red}}/X_{\text{red}}}$ under a morphism $(X', Z') \rightarrow (X, Z)$ of regular pairs of codimension c is $e \cdot s_{Z'_{\text{red}}/X'_{\text{red}}}$, where e denotes the ramification index, we obtain the following compatibility of purity isomorphisms:

Lemma 14. *Let $f : (X', Z') \rightarrow (X, Z)$ be a morphism of regular pairs of codimension c . Suppose that Z and Z' are irreducible and as cycles on X'_{red} we have $f_{\text{red}}^* Z_{\text{red}} = e \cdot Z'_{\text{red}}$ with a positive integer e (the ramification index). Then, for any $m \in \mathbb{N}$ invertible on X the following diagram commutes*

$$\begin{array}{ccc} H_Z^n(X, \mathbb{Z}/m\mathbb{Z}) & \xleftarrow[\text{purity}]{\sim} & H^{n-2c}(Z, \mathbb{Z}/m\mathbb{Z}(-c)) \\ \downarrow & & \downarrow \\ & & H^{n-2c}(Z', \mathbb{Z}/m\mathbb{Z}(-c)) \\ & & \downarrow \cdot e \\ H_{Z'}^n(X', \mathbb{Z}/m\mathbb{Z}) & \xleftarrow[\text{purity}]{\sim} & H^{n-2c}(Z', \mathbb{Z}/m\mathbb{Z}(-c)). \end{array}$$

Corollary 15. *Let X be a noetherian, regular scheme and $f : X' \rightarrow X$ a tamely ramified covering such that the branch locus $D \subseteq X$ is regular. Let Z be an irreducible component of D and let Z' denote its preimage in X' . Then,*

for any integer m dividing the ramification index of each irreducible component of Z' , the canonical map

$$H_Z^n(X, \mathbb{Z}/m\mathbb{Z}) \rightarrow H_{Z'}^n(X', \mathbb{Z}/m\mathbb{Z})$$

is the zero map for all $n \in \mathbb{N}$.

Proof. The surface X' and the underlying reduced subscheme of Z' are regular because the branch locus D is regular. Denote by Z'_k , $k = 1, \dots, r$ the irreducible components of Z' . For each k we can now apply Lemma 14 to the morphism

$$X' - \bigcup_{i \neq k} Z'_i \rightarrow X$$

to conclude that

$$H_Z^n(X, \mathbb{Z}/m\mathbb{Z}) \rightarrow H_{Z'_k}^n(X' - \bigcup_{i \neq k} Z'_i, \mathbb{Z}/m\mathbb{Z})$$

is the zero map. But

$$H_{Z'}^n(X', \mathbb{Z}/m\mathbb{Z}) = \bigoplus_k H_{Z'_k}^n(X' - \bigcup_{i \neq k} Z'_i, \mathbb{Z}/m\mathbb{Z}),$$

and the corollary follows. \square

Using Gabber's absolute purity theorem we can prove the following refined version of the proper base change theorem.

Lemma 16. *Let (X, Z) be a regular pair of codimension c and set $U = X - Z$. Let $\pi : X \rightarrow Y$ be a proper morphism such that Z_{red} intersects $(X_y)_{\text{red}}$ transversally for any closed point y of Y . Set $\Lambda = \mathbb{Z}/m\mathbb{Z}$ for an integer m prime to the residue characteristics of X . Then for any closed $y \in Y$ and any integer d the base change morphisms*

$$(R^n(\pi_U)_* \Lambda(d))_y \rightarrow H^n(U_y, \Lambda(d))$$

are isomorphisms for any $n \geq 0$.

Proof. Without loss of generality we may assume Y is the spectrum of a strictly henselian local ring with closed point y . Then, $\mu_m \cong \mathbb{Z}/m\mathbb{Z}$ on X and it suffices to prove the lemma for $d = 0$. We need to show that

$$H^n(U, \Lambda) \rightarrow H^n(U_y, \Lambda)$$

is an isomorphism. Consider the following diagram of excision sequences

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_Z^n(X, \Lambda) & \longrightarrow & H^n(X, \Lambda) & \longrightarrow & H^n(U, \Lambda) \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & H_{Z_y}^n(X_y, \Lambda) & \longrightarrow & H^n(X_y, \Lambda) & \longrightarrow & H^n(U_y, \Lambda) \longrightarrow \dots \end{array}$$

The homomorphisms $H^n(X, \Lambda) \rightarrow H^n(X_y, \Lambda)$ are bijective due to proper base change. Since by assumption Z_{red} intersects $(X_y)_{\text{red}}$ transversally, $(X_y, Z_y) \rightarrow (X, Z)$ is a morphism of regular pairs of codimension c yielding a commutative diagram

$$\begin{array}{ccc}
H_Z^n(X, \Lambda) & \xrightarrow{\sim} & H^{n-2c}(Z, \Lambda(-d)). \\
\downarrow & & \downarrow \\
H_{Z_y}^n(X_y, \Lambda) & \xrightarrow{\sim} & H^{n-2c}(Z_y, \Lambda(-d))
\end{array}$$

The horizontal maps are purity isomorphisms and the vertical map on the right is an isomorphism by proper base change. Hence, the vertical map on the left is an isomorphism and the lemma follows by applying the five lemma to the above diagram of exact sequences. \square

6 Cohomology and dual graphs

Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. In this section we relate the homology of the dual graph of D with the cohomology of X with support in D . Later, we will apply this in the situation where D is an exceptional fibre of a desingularized tame covering.

Proposition 17. *Let C be a projective curve over an algebraically closed field k with only ordinary double points and let Γ_C denote its dual graph. Let $S \subset C$ be a finite set of closed points containing the set C_{sing} of singular points of C . Define $C_N := \coprod_i C_i$, where C_i are the normalizations of the irreducible components of C . Set $S_N = S \times_C C_N$. For $m \in \mathbb{N}$ prime to the characteristic of k consider the homomorphisms of cohomology groups with coefficients in $\mathbb{Z}/m\mathbb{Z}$*

$$\begin{array}{ccccccc}
H^1(C_N - S_N) & \xrightarrow{\alpha} & H_{S_N}^2(C_N) & \xleftarrow[\text{purity}]{\sim} & H^0(S_N)(-1) & \xrightarrow{\text{norm}} & H^0(S)(-1), \\
\parallel & & & & & \nearrow & \\
H^1(C - S) & & & \xrightarrow{\beta} & & &
\end{array}$$

where α is the connecting homomorphism of the excision sequence associated to (C_N, S_N) . Then

$$\frac{\ker(\beta)}{\ker(\alpha)} \cong H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z}),$$

where $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ denotes singular homology with coefficients in $\mathbb{Z}/m\mathbb{Z}$.

Proof. The group $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ can be calculated using a cellular chain complex. The zero-skeleton $(\Gamma_C)_0$ consists of the nodes of the graph which correspond to the irreducible components C_i and the one-skeleton $(\Gamma_C)_1$ is all of Γ_C . Thus, the one-cells are the edges of the graph, which correspond to the points in C_{sing} . We give each edge s a direction by choosing an initial node $C_1(s)$ and an end node $C_2(s)$. Then $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ is the first homology of the sequence

$$0 \rightarrow H_1((\Gamma_C)_1, (\Gamma_C)_0, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{d} H_0((\Gamma_C)_0, \mathbb{Z}/m\mathbb{Z}) \rightarrow 0$$

and the map d can be identified with

$$\begin{aligned}
\bigoplus_{s \in C_{\text{sing}}} \mathbb{Z}/m\mathbb{Z} \cdot s &\rightarrow \bigoplus_i \mathbb{Z}/m\mathbb{Z} \cdot C_i. \\
s &\mapsto C_2(s) - C_1(s)
\end{aligned}$$

Let us now compute $\ker(\beta)/\ker(\alpha)$.

$$\begin{aligned}
\frac{\ker(\beta)}{\ker(\alpha)} &= \ker\left(\frac{H^1(C-S)}{\ker(\alpha)} \rightarrow H^0(S)(-1)\right) \\
&= \ker(\text{Im}(\alpha) \rightarrow H^0(S)(-1)) \\
&= \ker(\ker(H_{S_N}^2(C_N) \rightarrow H^2(C_N)) \rightarrow H^0(S)(-1)) \\
&= \ker(H_{S_N}^2(C_N) \rightarrow H^2(C_N)) \cap \ker(H_{S_N}^2(C_N) \rightarrow H^0(S)(-1)).
\end{aligned}$$

We identify the map $H_{S_N}^2(C_N) \rightarrow H^2(C_N)$ with

$$\begin{aligned}
\bigoplus_{s_N \in S_N} \mathbb{Z}/m\mathbb{Z} \cdot s_N &\rightarrow \bigoplus_i \mathbb{Z}/m\mathbb{Z} \cdot C_i, \\
s_N &\mapsto C(s_N)
\end{aligned}$$

where $C(s_N)$ is the component of C_N which contains s_N and $H_{S_N}^2(C_N) \rightarrow H^0(S)(-1)$ with

$$\bigoplus_{s_N \in S_N} \mathbb{Z}/m\mathbb{Z} \cdot s_N \rightarrow \bigoplus_{s \in S} \mathbb{Z}/m\mathbb{Z} \cdot s, \quad (1)$$

$$s_N \rightarrow s(s_N) \quad (2)$$

where $s(s_N)$ is the image of s_N in S . In particular, we obtain an isomorphism

$$\begin{aligned}
\bigoplus_{s \in C_{\text{sing}}} \mathbb{Z}/n\mathbb{Z} \cdot s &\rightarrow \ker\left(\bigoplus_{s_N \in S_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N \rightarrow \bigoplus_{s \in S} \mathbb{Z}/n\mathbb{Z} \cdot s\right), \\
s &\mapsto (s_N)_2(s) - (s_N)_1(s)
\end{aligned}$$

where $(s_N)_i(s) \in C_i(s)$ are the two preimages of s in C_N . Therefore, $\ker(\beta)/\ker(\alpha)$ is isomorphic to the kernel of the composition

$$\bigoplus_{s \in C_{\text{sing}}} \mathbb{Z}/m\mathbb{Z} \cdot s \rightarrow \bigoplus_{s_N \in S_N} \mathbb{Z}/m\mathbb{Z} \cdot s_N \rightarrow \bigoplus_i \mathbb{Z}/m\mathbb{Z} \cdot C_i,$$

which maps $s \in C_{\text{sing}}$ to $C_2(s) - C_1(s)$. Comparing with the calculation of $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ at the beginning of the proof we see that

$$\frac{\ker(\beta)}{\ker(\alpha)} \cong H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z}). \quad \square$$

Proposition 18. *Let X/B be an arithmetic surface and $D \subset X$ an snc-divisor. Let $S \subset D$ be a set of closed points containing the set D_{sing} of singular points of D . Denote by D_N the normalization of D and set $S_N = S \times_D D_N$. Then the following diagram of cohomology groups with coefficients in $\Lambda = \mathbb{Z}/m\mathbb{Z}$ (m prime to the residue characteristics of X) commutes*

$$\begin{array}{ccccc}
& & \delta & & \\
& \searrow & & \searrow & \\
H_{D-S}^3(X-S, \Lambda) & \longrightarrow & H^3(X-S, \Lambda) & \xrightarrow{\delta} & H_S^4(X, \Lambda) \\
\uparrow \text{purity} \sim & & & & \uparrow \text{purity} \sim \\
H^1(D-S, \Lambda(-1)) & & & & H^0(S, \Lambda(-2)) \\
\parallel & & & & \uparrow \text{norm} \\
H^1(D_N - S_N, \Lambda(-1)) & \xrightarrow{\delta} & H_{S_N}^2(D_N, \Lambda(-1)) & \xleftarrow[\text{purity}]{\sim} & H^0(S_N, \Lambda(-2)).
\end{array}$$

All maps δ are connecting homomorphisms of excision sequences.

Proof. Denote by D_i , $i = 1, \dots, r$ the irreducible components of D . Since

$$H_{D-S}^3(X-S, \Lambda) = \bigoplus_i H_{D_i-S}^3(X-S, \Lambda),$$

it suffices to prove the proposition for each component D_i separately. We may thus assume without loss of generality that D is a regular irreducible curve. In this case the above diagram reduces to

$$\begin{array}{ccccc} & & \delta & & \\ & \nearrow & & \searrow & \\ H_{D-S}^3(X-S, \Lambda) & \longrightarrow & H^3(X-S, \Lambda) & \xrightarrow{\delta} & H_S^4(X, \Lambda) \\ \uparrow \sim & & & & \uparrow \sim \\ & \delta & & & \\ H^1(D-S, \Lambda(-1)) & \longrightarrow & H_S^2(D, \Lambda(-1)) & \xleftarrow[\sim]{\text{purity}} & H^0(S, \Lambda(-2)). \end{array}$$

Consider the commutative diagram

$$\begin{array}{ccc} H_{D-S}^3(X-S, \Lambda) & \xrightarrow{\delta} & H_S^4(X, \Lambda) \\ \sim \uparrow & & \sim \uparrow \\ H^1(D-S, \underline{H}_{D-S}^2(X-S, \Lambda)) & \xrightarrow{\delta} & H_S^2(D, \underline{H}_D^2(X, \Lambda)) \\ \sim \downarrow & & \sim \downarrow \\ H^1(D-S, \Lambda(-1)) \otimes H_{D-S}^2(X-S, \Lambda(1)) & \xrightarrow{\delta \otimes res^{-1}} & H_S^2(D, \Lambda(-1)) \otimes H_D^2(X, \Lambda(1)) \\ \sim \uparrow \otimes s_{D-S/X-S} & & \sim \uparrow \otimes s_{D/X} \\ H^1(D-S, \Lambda(-1)) & \xrightarrow{\delta} & H_S^2(D, \Lambda(-1)). \end{array}$$

The restriction

$$res : H_D^2(X, \Lambda(1)) \rightarrow H_{D-S}^2(X-S, \Lambda(1))$$

is an isomorphism which maps the fundamental class $s_{D/X}$ to $s_{D-S/X-S}$. For this reason, the homomorphism $\delta \otimes res^{-1}$ in the third line of the diagram is well defined and the lowermost square commutes. Commutativity of the middle square follows because $\underline{H}_D(X)$ is a free sheaf which restricts to $\underline{H}_{D-S}(X-S)$ on $D-S$. The upper square commutes due to compatibility of the spectral sequences

$$\begin{aligned} H_S^i(D, \underline{H}_D^j(X, \Lambda)) &\Rightarrow H_S^{i+j}(X, \Lambda), \\ H^i(D-S, \underline{H}_{D-S}^j(X-S, \Lambda)) &\Rightarrow H_{D-S}^{i+j}(X-S, \Lambda). \end{aligned}$$

Furthermore, by [4], Proposition 1.2.1 the following diagram commutes

$$\begin{array}{ccc}
& H_S^4(X, \Lambda) & \\
& \uparrow \sim & \nwarrow \text{purity} \\
& H_S^2(D, \underline{H}_D^2(X, \Lambda)) & \\
& \downarrow \sim & \\
& H_S^2(D, \Lambda(-1)) \otimes H_D^2(X, \Lambda(1)) & \\
& \uparrow \sim \otimes_{S_D/X} & \\
H_S^2(D, \Lambda(-1)) & \xleftarrow{\sim \text{purity}} & H^0(S, \Lambda(-2)).
\end{array}$$

Putting the two diagrams together, the assertion of the proposition follows. \square

7 Setup and Notation

Throughout sections 8 and 9 we will stick to the following notation. We fix a proper arithmetic surface \bar{X}/B with geometric point $\bar{x} \rightarrow \bar{X}$ lying over a closed point $x \in \bar{X}$. We assume that the residue fields of B are either finite or algebraic closures of finite fields. Let $\bar{D} \subseteq \bar{X}$ be a tidy divisor whose support does not contain x . Let \bar{D}_h be the maximal subdivisor of \bar{D} with support on the isolated horizontal components of \bar{D} , i. e., on the horizontal components which do not intersect any other component. Set $X = \bar{X} - \bar{D}_h$ and $U = \bar{X} - \bar{D}$ and denote by $D \subseteq X$ the restriction of \bar{D} to X . We write D_v for the maximal vertical subdivisor of D and D_h for the maximal horizontal subdivisor, such that $D = D_v + D_h$. Notice that D_v is also the maximal vertical subdivisor of \bar{D} . The maximal horizontal subdivisor of \bar{D} is given by $\bar{D}_h + D_h$. Let W denote the union of all vertical prime divisors which are contained in a singular fibre of $\bar{X} \rightarrow B$ but not in \bar{D} . Put differently, W is the Zariski closure of the union of all reduced fibres $(U_b)_{red}$ where \bar{X}_b is singular. Denote by S the finite set of special points of \bar{D} , i. e., the set of singular points of \bar{D}_{red} .

Furthermore, we fix a full class of finite groups \mathfrak{c} whose orders are invertible on \bar{X} such that for all prime numbers $\ell \in \mathbb{N}(\mathfrak{c})$ we have $\mu_\ell \cong \mathbb{Z}/\ell\mathbb{Z}$ on X . Here, $\mathbb{N}(\mathfrak{c})$ denotes the submonoid of the positive integers of the orders of all groups in \mathfrak{c} . We choose an integer $m \in \mathbb{N}(\mathfrak{c})$ and set $\Lambda = \mathbb{Z}/m\mathbb{Z}$.

We denote by $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$ the category of all pointed desingularized \mathfrak{c} -coverings of (\bar{X}, \bar{D}) as defined at the end of Section 2. Viewing \bar{x} as geometric point of B we write $\mathfrak{I}_{B, \bar{x}}$ for the category of pointed finite étale \mathfrak{c} -coverings of B . By

$$(B' \rightarrow B) \mapsto ((\bar{X} \times_B B', \bar{D} \times_B B') \rightarrow (\bar{X}, \bar{D}))$$

$\mathfrak{I}_{B, \bar{x}}$ becomes a subcategory of $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$.

For $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ in $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$ let

$$\bar{X}' \rightarrow B' \rightarrow B$$

be the Stein factorization of $\bar{X}' \rightarrow \bar{X} \rightarrow B$. Then \bar{X}' is an arithmetic surface over B' . We use analogous notation for (\bar{X}', \bar{D}') as for (\bar{X}, \bar{D}) : We write U' for $\bar{X}' - \bar{D}$, \bar{D}'_h for the maximal subdivisor of \bar{D}' with support on the isolated horizontal components of \bar{D} and so on. Moreover, we write E' for the generalized exceptional divisor of $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$.

Lemma 19. *Let $Y' \rightarrow Y$ be a flat morphism of schemes which is locally of finite presentation. Let Z be a closed subscheme of Y and denote by Z' its preimage in Y' . Then every connected component of Z dominates a connected component of Z' .*

Proof. By [2], 2.4.6, the morphism $Y' \rightarrow Y$ is universally open. In particular, $Z' \rightarrow Z$ is open and thus $\text{Spec } \mathcal{O}_{Z',z'} \rightarrow \text{Spec } \mathcal{O}_{Z,z}$ is surjective for every point $z' \in Z'$ mapping to $z \in Z$. Suppose there is a connected component Z'_0 of Z' mapping nonsurjectively to a connected component Z_0 of Z . Then there are irreducible components Z_1 and Z_2 of Z_0 with nontrivial intersection such that Z_1 is contained in the image of Z'_0 and Z_2 is not. Let $z \in Z_0$ be a point in the intersection of Z_1 and Z_2 and $z' \in Z'_0$ a preimage of z . This produces a contradiction as $\text{Spec } \mathcal{O}_{Z',z'} \rightarrow \text{Spec } \mathcal{O}_{Z,z}$ is not surjective. \square

Lemma 20. *Let $\pi : (\bar{X}', \bar{D}') \rightarrow (\bar{X}_1, \bar{D}_1) \rightarrow (\bar{X}, \bar{D})$ be a desingularized \mathfrak{c} -covering. Then $\pi^* \bar{D}_h = \bar{D}'_h$ and D'_h is the horizontal part of $\pi^* D_h$.*

Proof. By the definition of \bar{D}_h and D_h it suffices to show that π maps the irreducible components of \bar{D}'_h to \bar{D}_h and those of D'_h to D_h . Using the generalized Abhyankar lemma ([6], Exp. XIII, 5.3.0) we convince ourselves that $X_1 \rightarrow X$ is flat. By Lemma 19 a connected component of \bar{D}_1 is mapped surjectively onto a connected component of \bar{D} . Furthermore, connected components of \bar{D}' are mapped surjectively onto connected components of \bar{D}_1 . Every irreducible component of \bar{D}'_h is thus mapped to an irreducible component of \bar{D}_h .

Since \bar{D}_h is regular and does not intersect the other components of \bar{D} , the tame covering $(\bar{X}_1, \bar{D}_1) \rightarrow (\bar{X}, \bar{D})$ has regular branch locus in a neighborhood of \bar{D}_h . By the generalized Abhyankar lemma (see [6], Exp. XIII, 5.3.0) the preimage $\bar{D}_{h,1}$ of \bar{D}_h in \bar{X}_1 is again regular and \bar{X}_1 is regular in a neighborhood of $\bar{D}_{h,1}$. In particular, $\bar{D}_{h,1}$ does not contain special points and thus $\pi^*(\bar{D}_h)$ is contained in \bar{D}'_h . Hence, the image of every irreducible component of D'_h is contained in D_h . \square

As a consequence of Lemma 20 we have $\pi^* X = X'$ and $\pi^* D_v = D'_v$. The preimage of D_h is the sum of D'_h and a divisor with support in E' . Our objective is to investigate whether U is $K(\pi, 1)$ with respect to \mathfrak{c} .

8 Killing cohomology with support

In order to prove that U is $K(\pi, 1)$ with respect to a full class of finite groups \mathfrak{c} it is necessary and sufficient to show that for any $\Lambda \in \mathfrak{c}$ of the form $\Lambda = \mathbb{Z}/m\mathbb{Z}$ and any $n \geq 1$ we have

$$\varinjlim_{U' \rightarrow U} H^n(U', \Lambda) = 0,$$

where the limit runs over all finite étale \mathfrak{c} -coverings of U (see [11], Proposition 2.1). This amounts to the same as to take the limit over $\mathfrak{I}_{X,D,\bar{x}}$ instead. In this section we give a criterion for the restriction

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(X', \Lambda) \longrightarrow \varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(U', \Lambda)$$

to be surjective for $n \geq 2$.

Lemma 21. *Let $Z \leq D_v$ be a rational subdivisor (i. e., Z is a rational curve) and let V be an open subscheme of X . Suppose that for every geometric point \bar{b} above a closed point $b \in B$ the natural map*

$$\pi_1(b, \bar{b})(c) \rightarrow \pi_1(B, \bar{b})(c)$$

is injective. Then the restriction

$$\varinjlim_{\mathfrak{I}_{B, \bar{x}}} H^2(V' - S', \Lambda) \rightarrow \varinjlim_{\mathfrak{I}_{B, \bar{x}}} H^2(V' - Z' \cup S', \Lambda)$$

is surjective. (Remember that S' is the set of special points of \bar{D}' .)

Proof. We have to show that for B' in $\mathfrak{I}_{B, \bar{x}}$ the cokernel of $H^2(V' - S', \Lambda) \rightarrow H^2(V' - Z' \cup S', \Lambda)$ vanishes in the limit over $\mathfrak{I}_{B, \bar{x}}$. Since a base change to B' preserves the assumptions of the lemma, we may assume that $B' = B$. Furthermore, we may assume that $V \cap Z$ is dense in Z . Otherwise, Z has irreducible components in the complement of V , which we can remove without changing the above cohomology groups. Denote by T the union of S with the finite set of closed points $Z - V$. Then $V - S = V - T$.

By proposition 18 we have the following commutative diagram

$$\begin{array}{ccccc} H^2(V - Z \cup T, \Lambda) & \longrightarrow & H^3_{Z-T}(V - T, \Lambda) & \longrightarrow & H^3(V - T, \Lambda) \\ & & \uparrow \sim & & \downarrow \\ & & H^1(Z - T, \Lambda(-1)) & \xrightarrow{\beta(-1)} & H^0(T, \Lambda(-2)), \end{array}$$

where $\beta(-1)$ is the (-1) -twist of the map β defined in proposition 17. It thus suffices to show that the kernel of β vanishes in the limit over $\mathfrak{I}_{B, \bar{x}}$. Without loss of generality we may assume that Z is contained in a single closed fibre of $X \rightarrow B$ over some point $b \in B$ with residue field $k(b)$. Let $\bar{k}(b)$ be an algebraic closure of $k(b)$ and denote by \bar{Z} and \bar{T} the base change of Z and T , respectively, to $\bar{k}(b)$. Moreover, write Z_N for the normalization of Z and \bar{Z}_N for its base change to $\bar{k}(b)$. Consider the diagram of cohomology groups with coefficients in Λ

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & H^1(\bar{Z}_N)^{G_{k(b)}} & \longrightarrow & H^1(\bar{Z} - \bar{T})^{G_{k(b)}} & \xrightarrow{\bar{\beta}} & H^0(\bar{T})(-1)^{G_{k(b)}} \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H^1(Z_N) & \longrightarrow & H^1(Z - T) & \xrightarrow{\beta} & H^0(T)(-1), \\ & & \uparrow & & \uparrow & & \\ & & H^1(k(b))^d & \xrightarrow{=} & H^1(k(b))^d & & \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array}$$

where d is the number of components of Z_N . The vertical sequences are induced by the Hochschild-Serre spectral sequences

$$\begin{aligned} H^i(k(b), H^j(\bar{Z}_N), \Lambda) &\Rightarrow H^{i+j}(Z_N, \Lambda), \\ H^i(k(b), H^j(\bar{Z}_N - \bar{T}_N), \Lambda) &\Rightarrow H^{i+j}(Z_N - T_N, \Lambda). \end{aligned}$$

The upper horizontal sequence is exact by the following reason: According to proposition 17, the first homology group $H_1(\Gamma_Z, \Lambda)$ of the dual graph Γ_Z of \bar{Z} is isomorphic to $\ker(\bar{\beta})/\ker(\bar{\alpha})$, where $\bar{\alpha}$ denotes the connecting homomorphism of the excision sequence associated to $\bar{T}_N \hookrightarrow \bar{Z}_N$. Since we assumed Z to be rational, Γ_Z is a tree, and thus its first homology group vanishes. It follows that the kernel of $\bar{\beta}$ equals the image of the map

$$\gamma : H^1(\bar{Z}_N, \Lambda) \hookrightarrow H^1(\bar{Z}_N - \bar{T}_N, \Lambda) = H^1(\bar{Z} - \bar{T}, \Lambda).$$

Taking $G_{k(b)}$ -invariants we obtain the upper sequence of the above diagram which is therefore exact. A diagram chase now shows the exactness of the lower horizontal sequence.

Again by the rationality assumption on Z , the cohomology group $H^1(\bar{Z}_N)$ vanishes. The above diagram shows that the kernel of β equals $H^1(k(b))^d$. By the assumption on fundamental groups it vanishes in the limit over $\mathfrak{I}_{B, \bar{x}}$. \square

Proposition 22. *Suppose that the following conditions are satisfied:*

- (i) (\bar{X}, \bar{D}) has enough tame coverings.
- (ii) Every connected component of D has at least one horizontal component.
- (iii) For every geometric point \bar{b} above a closed point $b \in B$ the natural map

$$\pi_1(b, \bar{b})(\mathfrak{c}) \rightarrow \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective.

Then, for any $n \geq 2$ the restriction

$$\varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(X', \Lambda) \rightarrow \varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(U', \Lambda)$$

is surjective.

Proof. Let (\bar{X}', \bar{D}') be an object of $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$ and φ an element of $H^n(U', \Lambda)$. We have to show that there is a desingularized \mathfrak{c} -covering $(\bar{X}'', \bar{D}'') \rightarrow (\bar{X}', \bar{D}')$ such that the image of φ in $H^n(U'', \Lambda)$ lifts to $H^n(X'', \Lambda)$. Since the assumptions are stable under desingularized \mathfrak{c} -coverings by Lemma 20 and Lemma 10, we may assume $(\bar{X}', \bar{D}') = (\bar{X}, \bar{D})$. We first construct (\bar{X}', \bar{D}') in $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$ such that the image of φ in $H^n(U', \Lambda)$ lifts to $H^n(X' - S', \Lambda)$.

Let us treat the case $n = 2$. Since (\bar{X}, \bar{D}) has enough tame coverings, there is a desingularized \mathfrak{c} -covering

$$(X', D') \rightarrow (X_1, D_1) \rightarrow (X, D),$$

such that m divides the ramification index of each irreducible component of D_1 . We have the following commutative diagram of excision sequences with coefficients Λ :

$$\begin{array}{ccccc} H^2(X - S) & \longrightarrow & H^2(U) & \longrightarrow & H^3_{D-S}(X - S) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(X' - S' \cup E') & \longrightarrow & H^2(U') & \longrightarrow & H^3_{D'-S' \cup E'}(X' - S' \cup E'). \end{array}$$

Let φ' denote the image of φ in $H^2(U', \Lambda)$. By proposition 15 φ' is mapped to zero in $H_{D' - S' \cup E'}^3(X' - S' \cup E', \Lambda)$. Hence, there is $\varphi'_1 \in H^2(X' - S' \cup E', \Lambda)$ mapping to φ' . Since E' is rational by Proposition 5, we can apply Lemma 21 with $V = X' - S'$ and $Z = E'$ to obtain a finite étale \mathfrak{c} -covering $B'' \rightarrow B'$ and thus via base change a finite étale \mathfrak{c} -covering $\bar{X}'' \rightarrow \bar{X}'$ such that the image of φ'_1 in $H^2(X'' - (S'' \cup E'), \Lambda)$ lies in the image of

$$H^2(X'' - S'', \Lambda) \rightarrow H^2(X'' - S'' \cup E'', \Lambda).$$

It thus can be lifted to an element $\varphi''_2 \in H^2(X'' - S'', \Lambda)$. Changing notation we may assume that φ lifts to $\varphi_2 \in H^2(X - S, \Lambda)$.

Now assume that $n \geq 3$. By the same argument as for $n = 2$ there is a desingularized \mathfrak{c} -covering $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ such that the image of φ in $H^n(U', \Lambda)$ lifts to $H^n(X' - (S' \cup E'))$. In particular, it lifts to $H^n(X' - D'_v, \Lambda)$ and thus we may assume that φ lifts to $H^n(X - D_v, \Lambda)$ right away. Consider the excision sequence

$$\dots \rightarrow H^n(X - S, \Lambda) \rightarrow H^n(X - D_v, \Lambda) \rightarrow H_{D_v - S}^{n+1}(X - S, \Lambda) \rightarrow \dots$$

By purity we have

$$H_{D_v - S}^{n+1}(X - S, \Lambda) \cong H^{n-1}(D_v - S, \Lambda(-1)).$$

For each component Z_i of D_v lying over a closed point $b_i \in B$ with geometric point \bar{b}_i consider the Hochschild-Serre spectral sequence

$$H^r(b_i, H^s(Z_{i, \bar{b}_i} - S_{\bar{b}_i}, \Lambda)) \Rightarrow H^{r+s}(Z_i - S, \Lambda).$$

For any i $Z_{i, \bar{b}_i} - S_{\bar{b}_i}$ is an affine curve over an algebraically closed field and $k(b_i)$ is a finite field. Therefore, they both have cohomological dimension less or equal to one. We conclude that

$$H^{n-1}(Z_i - S, \Lambda) = 0$$

for $n \geq 4$ and

$$H^2(Z_i - S, \Lambda) \cong H^1(b_i, H^1(Z_{i, \bar{b}_i} - S_{\bar{b}_i}, \Lambda)).$$

By the assumption on fundamental groups there is a finite étale \mathfrak{c} -covering $B'_i \rightarrow B$ and thus via base change a finite étale \mathfrak{c} -covering $\bar{X}'_i \rightarrow \bar{X}$ such that

$$H^1(b_i, H^1(Z_{i, \bar{b}_i} - S_{\bar{b}_i}, \Lambda)) \rightarrow H^1(b_i \times_B B'_i, H^1(Z_{i, \bar{b}_i} - S_{\bar{b}_i}, \Lambda))$$

is the zero map. Let B' be the compositum of all extensions B'_i . By compatibility with the Hochschild-Serre spectral sequence and the purity isomorphism we conclude that φ_1 maps to 0 in $H_{D'_v - S'}^4(X' - S', \Lambda)$. As before we replace \bar{X} by \bar{X}' and may assume that φ_1 maps to 0 in $H_{D_v - S}^4(X - S, \Lambda)$. Hence, φ_1 lifts to $\varphi_2 \in H^3(X - S, \Lambda)$.

Having lifted φ to $\varphi_2 \in H^n(X - S, \Lambda)$ for any $n \geq 2$ we now construct (\bar{X}', \bar{D}') in $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$ such that φ_2 lifts to $H^n(X, \Lambda)$. Consider the excision sequence:

$$\dots \longrightarrow H^n(X, \Lambda) \longrightarrow H^n(X - S, \Lambda) \longrightarrow H_S^{n+1}(X, \Lambda) \longrightarrow \dots$$

The cohomology group $H_S^{n+1}(X, \Lambda)$ is the direct sum of all $H_s^{n+1}(X, \Lambda)$ for the finitely many points $s \in S$. For $s \in S$ choose an irreducible component D_s of D passing through s . Since (\bar{X}, \bar{D}) has enough tame coverings, we find a desingularized \mathfrak{c} -covering $(\bar{X}', \bar{D}') \rightarrow (\bar{X}_1, \bar{D}_1) \rightarrow (\bar{X}, \bar{D})$ such that m divides the ramification indices of all irreducible components of \bar{D}_1 lying over D_s and is unramified in all other prime divisors passing through s . Since the branch locus is regular in a neighborhood of s , the pair (\bar{X}_1, \bar{D}_1) is regular at all preimage points s'_1, \dots, s'_r of s . Hence, $\bar{X}' \rightarrow \bar{X}_1$ is an isomorphism in a neighborhood of s'_1, \dots, s'_r . Therefore, by Corollary 15, the homomorphism

$$H_S^{n+1}(X, \Lambda) \rightarrow \bigoplus_i H_{s'_i}^{n+1}(X', \Lambda)$$

is the zero map. Take a desingularized \mathfrak{c} -covering $(\bar{X}'', \bar{D}'') \rightarrow (\bar{X}, \bar{D})$ dominating the coverings $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ constructed for each $s \in S$. We obtain a diagram of excision sequences

$$\begin{array}{ccccc} H^n(X, \Lambda) & \longrightarrow & H^n(X - S, \Lambda) & \longrightarrow & H_S^{n+1}(X, \Lambda) \\ \downarrow & & \downarrow & & \downarrow \\ H^n(X'', \Lambda) & \longrightarrow & H^n(X'' - (S'' \cup E''), \Lambda) & \longrightarrow & H_{S'' \cup E''}^{n+1}(X'', \Lambda) \end{array}$$

The homomorphism

$$H_S^{n+1}(X, \Lambda) \rightarrow H_{S'' \cup E''}^{n+1}(X'', \Lambda)$$

is the zero map. This implies the assertion. \square

9 Killing the Cohomology of higher direct images

In section 8 we established the surjectivity of

$$\varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(X', \Lambda) \rightarrow \varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(U', \Lambda).$$

In this section we show that if X/B is of local type,

$$\varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(X', \Lambda) = 0$$

for $n \geq 3$. In case $n = 2$ there are additional problems to be dealt with. These will occupy most of this section.

Lemma 23. *Suppose that X/B is of local type. Then*

$$\varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(X', \Lambda) = 0$$

for $n \geq 3$ and

$$\varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} \ker(H^2(X', \Lambda) \xrightarrow{edge} H^0(B', R^2 \pi'_* \Lambda)) = 0.$$

Proof. Let \bar{b} be a geometric point of B . Consider the projections $\pi' : X' \rightarrow B'$ for (X', D') in $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$. By Lemma 16 the stalk of $R^j \pi'_* \Lambda$ at \bar{b} is isomorphic to $H^j(X'_b, \Lambda)$. Since X'_b is a curve over an algebraically closed field, it has cohomological dimension less or equal to 2. Therefore, $R^j \pi'_* \Lambda = 0$ for $j \geq 3$. Since

$$H^i(B', R^j \pi'_* \Lambda) \cong H^i(b', R^j \pi'_* \Lambda) \cong H^i(k(b'), H^j(X'_b, \Lambda)),$$

by Lemma 16, there is an étale \mathfrak{c} -covering $B'' \rightarrow B'$ such that

$$H^1(B', R^j \pi'_* \Lambda) \rightarrow H^1(B'', R^j \pi''_* \Lambda)$$

is the zero map for all $j \geq 0$. The result now follows from an inspection of the Leray spectral sequences

$$H^i(B', R^j \pi'_* \Lambda) \Rightarrow H^{i+j}(X', \Lambda).$$

for each $X' \in \mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$ and taking the limit. \square

In order to complete the proof for $n = 2$ we have to deal with the cokernel of

$$H_D^2(X, \Lambda) \rightarrow H^2(X, \Lambda).$$

In the following we explain how to relate this homomorphism with the intersection matrix of the irreducible components of the singular fibres.

Lemma 24. *Suppose that B is strictly henselian with closed point s . Denote by ρ the intersection matrix of the components of the special fibre of $\bar{\pi} : \bar{X} \rightarrow B$. Then, for any integer c the following diagram commutes*

$$\begin{array}{ccc} H_{D_v}^2(X, \Lambda(c+1)) & \longrightarrow & H^2(X, \Lambda(c+1)) \\ \text{purity} \uparrow \sim & & \sim \downarrow \text{base change} \\ H^0(D_v, \Lambda(c)) & & H^2(X_s, \Lambda(c+1)) \\ \uparrow \sim & & \sim \downarrow \text{deg} \\ \bigoplus_{C \subseteq D_v} \Lambda(c) \cdot C & \xrightarrow{\rho} & \bigoplus_{C \subseteq \bar{X}_s, C \cap \bar{D}_h = \emptyset} \Lambda(c) \cdot C. \end{array}$$

Proof. It suffices to prove the lemma for $c = 0$. Consider the commutative diagram

$$\begin{array}{ccccc} H_{D_v}^2(X, \mu_m) & \longrightarrow & H^2(X, \mu_m) & \xrightarrow{\sim} & H^2(X_s, \mu_m) \\ \sim \uparrow & & \sim \uparrow & & \sim \uparrow \\ \bigoplus_{C \subseteq D_v} H_C^1(X, \mathbb{G}_m) \otimes \Lambda & \longrightarrow & \text{Pic}(X) \otimes \Lambda & \longrightarrow & \bigoplus_{C \cap \bar{D}_h = \emptyset} \text{Pic}(C) \otimes \Lambda \\ \sim \uparrow & & & & \sim \downarrow (\text{deg}_C)_C \\ \bigoplus_{C \subseteq D_v} \Lambda \cdot C & \longrightarrow & & \longrightarrow & \bigoplus_{C \cap \bar{D}_h = \emptyset} \Lambda \cdot C. \end{array}$$

The direct sums on the right hand side run only over irreducible components of X_s with trivial intersection with \bar{D}_h as these are precisely the components of X_s which are proper over B . The upper right horizontal isomorphism comes from Lemma 16. The upper vertical maps are connecting homomorphisms of the Kummer sequence. The concatenation of the left hand side vertical arrows yields the purity isomorphism and the right hand vertical arrows give the degree map on $H^2(X_s, \mu_m)$. The restrictions

$$\text{Pic}(X) \rightarrow \text{Pic}(C)$$

are given by $D \mapsto D \cdot C$ where $D \cdot C$ denotes the intersection product of the divisor D with the curve C . Composition with \deg_C yields the intersection number $(D \cdot C)$. We conclude that the lower horizontal map is indeed given by the intersection matrix $\rho_{C_1, C_2} = (C_1 \cdot C_2)$. \square

We set

$$\mathbb{Z}(\mathfrak{c}) = \varprojlim_{n \in \mathbb{N}(\mathfrak{c})} \mathbb{Z}/n\mathbb{Z} = \prod_{\ell \in \mathbb{N}(\mathfrak{c}) \text{ prime}} \mathbb{Z}_\ell.$$

Lemma 25. *Assume that (\bar{X}, \bar{D}) has enough tame coverings. Then, for every integer $d \in \mathbb{N}(\mathfrak{c})$ there is a desingularized \mathfrak{c} -covering $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ such that the image of*

$$H_D^2(\bar{X}, \mathbb{Z}(\mathfrak{c})(1)) \rightarrow H_{D'}^2(\bar{X}', \mathbb{Z}(\mathfrak{c})(1))$$

is divisible by d .

Proof. By purity we have

$$\bigoplus_{C \subseteq \bar{D}} \mathbb{Z}(\mathfrak{c}) \cdot C \xrightarrow{\sim} H_D^2(\bar{X}, \mathbb{Z}(\mathfrak{c})(1)).$$

Moreover, if $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ is a desingularized \mathfrak{c} -covering, the induced map

$$\bigoplus_{C \subseteq \bar{D}} \mathbb{Z}(\mathfrak{c}) \cdot C \rightarrow \bigoplus_{C' \subseteq \bar{D}'} \mathbb{Z}(\mathfrak{c}) \cdot C'$$

is given by the pull-back of divisors. Let $C \subseteq \bar{D}$ be an irreducible component and $c \in C$ a closed point of \bar{D} . Since (\bar{X}, \bar{D}) has enough tame coverings, there is $f_c \in K(\bar{X})^\times$ such that $\deg_C(f_c) = m_c > 0$ and $\deg_Z(f_c) = 0$ for all other irreducible components Z of \bar{D} passing through c . Hence, in a Zariski neighborhood U_c of c we have $\text{div } f_c = m_c C$. Denote by m'_c the maximal factor of m_c contained in $\mathbb{N}(\mathfrak{c})$. Let $\phi_c : (\bar{X}_c, \bar{D}_c) \rightarrow (\bar{X}, \bar{D})$ be a desingularized \mathfrak{c} -covering with function field extension

$$K(\bar{X}_c) = K(\bar{X})[{}^{m'_c d} \sqrt{f_c}] | K(\bar{X}).$$

Then $\text{div } f_c \subseteq \bar{X}_c$ is divisible by $m'_c d$. Thus, $\phi_c^*(C) \cap \phi_c^{-1}(U_c)$ is divisible by d , i. e., the coefficients of all irreducible components of $\phi_c^*(C)$ whose generic points lie over U_c are divisible by d . This property is conserved by further desingularized coverings.

There are finitely many closed points $c_1, \dots, c_n \in C$ such that the open subschemes U_{c_1}, \dots, U_{c_n} cover C . Let $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ be a desingularized \mathfrak{c} -covering dominating all coverings $(\bar{X}_{c_i}, \bar{D}_{c_i}) \rightarrow (\bar{X}, \bar{D})$ constructed above. Then the pullback of C to \bar{X}' is divisible by d . \square

Corollary 26. *Let B_0 be the strict henselization of B at a geometric point of B over a closed point. Denote by X_0 and D_0 the base change of X and D , respectively, to B_0 . Assume that \bar{D}_h is nonempty and meets all irreducible components of W . If (\bar{X}, \bar{D}) has enough tame coverings, the cokernel of*

$$H_{D_0}^2(X_0, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \rightarrow H^2(X_0, \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

vanishes in the limit over $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$.

Proof. We may replace B by B_0 . We just have to check that all tame coverings of (\bar{X}_0, \bar{D}_0) occuring in the proof come from coverings of (\bar{X}, \bar{D}) . It suffices to prove that the cokernel of

$$\phi : H_{D_v}^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \rightarrow H^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

vanishes in the limit over $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$ as $H_{D_v}^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$ is a direct summand of $H_D^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$. Taking the inverse limit over all $\Lambda \cong \mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}(\mathfrak{c})$ of the diagrams in Lemma 24 and setting $c = 0$, we obtain

$$\begin{array}{ccc} H_{D_v}^2(X, \mathbb{Z}(\mathfrak{c})(1)) & \xrightarrow{\phi} & H^2(X, \mathbb{Z}(\mathfrak{c})(1)) \\ \text{purity} \uparrow \sim & & \sim \downarrow \text{base change} \\ H^0(D_v, \mathbb{Z}(\mathfrak{c})) & & H^2(X_s, \mathbb{Z}(\mathfrak{c})(1)) \\ \uparrow \sim & & \sim \downarrow \text{deg} \\ \bigoplus_{C \subseteq D_v} \mathbb{Z}(\mathfrak{c}) \cdot C & \xrightarrow{\rho} & \bigoplus_{C \subseteq D_v} \mathbb{Z}(\mathfrak{c}) \cdot C. \end{array}$$

Since we assumed that \bar{D}_h meets all irreducible components of W , we have that $C \cap \bar{D}_h = \emptyset$ if and only if $C \subseteq D_v$. By [10], Theorem 9.1.23 the intersection matrix of the components of the special fibre is negative semidefinite and its radical is generated by the special fibre. Since we assumed that \bar{D}_h is nonempty, the support of D does not comprise all irreducible components of the special fibre. Hence, the restriction ρ of the intersection matrix to the components of D_v is negative definite. We conclude that

$$\phi \otimes \mathbb{Q} : H_D^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \otimes \mathbb{Q} \rightarrow H^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \otimes \mathbb{Q}$$

is an isomorphism and thus the cokernel of ϕ is \mathfrak{c} -torsion. Take $d \in \mathbb{N}(\mathfrak{c})$ such that $d \cdot \text{coker } \phi = 0$. By Lemma 25 there is a desingularized \mathfrak{c} -covering $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ of (\bar{X}, \bar{D}) such that the image of

$$H_D^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \rightarrow H_{D'}^2(X', \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

is divisible by d . Taking into account that multiplication by d is injective on $H^2(X', \hat{\mathbb{Z}}(\mathfrak{c})(1))$ this proves the result. \square

Proposition 27. *Assume that \bar{D}_h is nonempty and intersects all irreducible components of W . Let ϕ be in the image of*

$$H^2(X, \Lambda) \rightarrow H^2(U, \Lambda).$$

Assume further that (\bar{X}, \bar{D}) has enough tame coverings. Then there is a desingularized \mathfrak{c} -covering $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ such that the image of ϕ in $H^2(U', \Lambda)$ can be lifted to an element ψ' of $H^2(X', \Lambda)$, which lies in the kernel of the edge homomorphism

$$H^2(X', \Lambda) \rightarrow H^0(B', R^2\pi_*\Lambda).$$

Proof. Consider the diagram

$$\begin{array}{ccccc} H_D^2(X, \Lambda) & \longrightarrow & H^2(X, \Lambda) & \longrightarrow & H^2(U, \Lambda) \\ \sim \downarrow \text{edge} & & \downarrow \text{edge} & & \\ H^0(B, R_D^2\pi_*\Lambda) & \longrightarrow & H^0(B, R^2\pi_*\Lambda) & & \end{array}$$

The left vertical arrow is an isomorphism because due to purity $R_D^j\pi_*\Lambda = 0$ for $j \leq 1$. We conclude that it suffices to show that the cokernel of

$$H^0(B, R_D^2\pi_*\Lambda) \rightarrow H^0(B, R^2\pi_*\Lambda)$$

vanishes in the limit over $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$. We have a direct sum decomposition indexed by the irreducible components D_i of D

$$R_D^2\pi_*\Lambda = \bigoplus_i R_{D_i}^2\pi_*\Lambda.$$

It is sufficient to prove that the cokernel of the vertical part vanishes after a desingularized \mathfrak{c} -covering: Both $R_{D_v}^2\pi_*\Lambda$ and $R^2\pi_*\Lambda$ are skyscraper sheaves with support in the singular locus of $X \rightarrow B$. We can treat each singular fibre separately and thus assume that B is a henselian discrete valuation ring. We only have to make sure that the constructed desingularized \mathfrak{c} -covering extends to a desingularized \mathfrak{c} -covering above the initial base scheme. We have the following diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(B, R_{D_v}^2\pi_*\mathbb{Z}(\mathfrak{c})) & \xrightarrow{\cdot m} & H^0(B, R_{D_v}^2\pi_*\mathbb{Z}(\mathfrak{c})) & \longrightarrow & H^0(B, R_{D_v}^2\pi_*\Lambda) \longrightarrow 0 \\ & & \downarrow \rho & & \downarrow \rho & & \downarrow \\ 0 & \longrightarrow & H^0(B, R^2\pi_*\mathbb{Z}(\mathfrak{c})) & \xrightarrow{\cdot m} & H^0(B, R^2\pi_*\mathbb{Z}(\mathfrak{c})) & \longrightarrow & H^0(B, R^2\pi_*\Lambda) \longrightarrow 0. \end{array} \quad (3)$$

The exactness of the above sequences can be checked by using the explicit description of the cohomology groups involved.

In order to show that the cokernel of the right hand side vertical map in the diagram (3) vanishes after a desingularized \mathfrak{c} -covering it suffices to show that the cokernel of the middle vertical map does so. The stalk of the morphism $R_{D_v}^2\pi_*\hat{\mathbb{Z}}(\mathfrak{c}) \rightarrow R^2\pi_*\hat{\mathbb{Z}}(\mathfrak{c})$ at \bar{b} is

$$H_{D_{\bar{b}}}^2(X^{sh}, \hat{\mathbb{Z}}(\mathfrak{c})) \rightarrow H^2(X^{sh}, \hat{\mathbb{Z}}(\mathfrak{c})).$$

By Lemma 24 it is given by the intersection matrix ρ of the components of $D_{\bar{b}}$. Since $D_{\bar{b}}$ does not contain all components of the geometric special fibre, ρ is injective. Denote by \mathcal{F} the cokernel. By Corollary 26 there is a desingularized \mathfrak{c} -covering $(\bar{X}', \bar{D}') \rightarrow (\bar{X}, \bar{D})$ such that $\mathcal{F} \rightarrow \mathcal{F}'$ is the zero map (where \mathcal{F}' is the respective cokernel defined on X'). We have an exact sequence

$$0 \rightarrow H^0(B, R_{D_v}^2\pi_*\hat{\mathbb{Z}}(\mathfrak{c})) \rightarrow H^0(B, R^2\pi_*\hat{\mathbb{Z}}(\mathfrak{c})) \rightarrow H^0(\mathcal{G}, \mathcal{F}).$$

So the cokernel of $H^0(B, R_{D_v}^2 \pi_* \hat{\mathbb{Z}}(\mathfrak{c})) \rightarrow H^0(B, R^2 \pi_* \hat{\mathbb{Z}}(\mathfrak{c}))$ is a subgroup of \mathcal{F} . This shows the result. \square

10 Neighborhoods with enough tame coverings

For the construction of étale neighborhoods with enough tame coverings we need the following variant of prime evasion.

Lemma 28. *Let r and s be positive integers. Let A be a noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ prime ideals such that for $i \neq j$ \mathfrak{q}_i is not contained in \mathfrak{q}_j . For $j \leq s$ define the integer m_j by*

$$\mathfrak{p}_1 \cdot \dots \cdot \mathfrak{p}_r \subseteq \mathfrak{q}_j^{m_j} \setminus \mathfrak{q}_j^{m_j+1}.$$

Then there is $a \in A$ such that for $i \leq r$ $a \in \mathfrak{p}_i$ and for $j \leq s$ $a \notin \mathfrak{q}_j^{m_j+1}$.

For the rest of this section we use the following notation: For an integral closed subscheme Z of an affine scheme $\text{Spec } A$ we denote by \mathfrak{p}_Z the prime ideal of A corresponding to the generic point of Z . Moreover, we write $m_x(Z)$ for the multiplicity of a closed subscheme Z in a point x .

Lemma 29. *Let X/B be an arithmetic surface such that B is local with generic point η isomorphic to the spectrum of a global field (i. e., B is the localization of a global Dedekind scheme at a closed point). Let x_1, \dots, x_n be finitely many points of X . Then there are horizontal prime divisors $G_1, \dots, G_s, G_{s+1}, \dots, G_r$ such that G_1, \dots, G_s and G_{s+1}, \dots, G_r each generate the Weil divisor class group $CH^1(X)$ of X . Furthermore, the supports of G_i for $i = 1, \dots, r$ do not contain x_j for $j = 1, \dots, n$ and the supports of G_i and G_j for $i \leq s$ and $j > s$ are disjoint.*

Proof. The generic fibre X_η of $X \rightarrow B$ is a smooth curve over a global field. By the Mordell-Weil theorem (see [12]) its Weil divisor class group is finitely generated. Denote by C_1, \dots, C_l the irreducible components of the special fibre. The Weil divisor class group of X is generated by the Weil divisor class group of X_η and by C_1, \dots, C_l . It is therefore also finitely generated, by prime divisors D_1, \dots, D_m , say.

Since X is quasi-projective over an affine scheme, there is an affine open subscheme $\text{Spec } A \subseteq X$ containing x_1, \dots, x_n , as well as the generic points of D_1, \dots, D_m and of C_1, \dots, C_l (see [10], Proposition 3.3.36). By Lemma 28 we can choose $f_1, \dots, f_m \in A$ such that for $i = 1, \dots, m$

$$\begin{aligned} f_i &\in \mathfrak{p}_{D_i} \setminus \mathfrak{p}_{D_i}^2, \\ f_i &\notin \mathfrak{p}_{C_j}^{m_{C_j}(D_i)+1} \text{ for } j = 1, \dots, l, \\ f_i &\notin \mathfrak{p}_{x_j}^{m_{x_j}(D_i)+1} \text{ for } j = 1, \dots, n. \end{aligned}$$

Viewing f_i as elements of $K(X)^\times$ we obtain divisors $D_1 - \text{div } f_1, \dots, D_m - \text{div } f_m$ generating the Weil divisor class group. The supports of the divisors $D_i - \text{div } f_i$ do not contain x_1, \dots, x_n and the coefficients of C_1, \dots, C_l are zero, i. e., $D_i - \text{div } f_i$ are horizontal. Denote by G_1, \dots, G_s the prime divisors in the support

of $D_1 - \text{div } f_1, \dots, D_m - \text{div } f_m$. Then G_1, \dots, G_s are horizontal prime divisors generating the Weil divisor class group whose supports do not contain x_1, \dots, x_n .

Denote by z_1, \dots, z_t the intersection points of G_1, \dots, G_s with the special fibre. By the same argument as above we find horizontal prime divisors G_{s+1}, \dots, G_r generating $\text{CH}^1(X)$ whose supports do not contain x_1, \dots, x_n nor z_1, \dots, z_t . Hence, the support of G_i for $i \leq s$ is disjoint from the support of G_j for $j > s$ as z_1, \dots, z_t are the only possible intersection points of G_i with another divisor. \square

Lemma 30. *Let X/B be an arithmetic surface and let G_1, \dots, G_s be horizontal prime divisors generating the Weil divisor class group $\text{CH}^1(X)$ of X . Let x be a closed point of X such that X is regular at x and x is not contained in any G_j for $j = 1, \dots, s$. Denote by $X' \rightarrow X$ the blowup of X in x . Let G be a horizontal prime divisor disjoint from G_1, \dots, G_s with nontrivial intersection with the exceptional locus E . Then G_1, \dots, G_s, G generate $\text{CH}^1(X') \otimes \mathbb{Q}$.*

Proof. The Weil divisor class group of X' is generated by G_1, \dots, G_s and E . Let G_0 denote the image of G in X . Since G_1, \dots, G_s generate the Weil divisor class group of X , there are $n_j \in \mathbb{Z}$ such that

$$G_0 = \sum_{j=1}^s n_j G_j$$

in $\text{CH}^1(X)$. By [10], Chapter 9, Proposition 2.23 the pullback of G_0 to X' is given by

$$G + m_x(G_0) \cdot E.$$

Since $x \in G_0$, the multiplicity $m_x(G_0)$ is positive. In $\text{CH}^1(X') \otimes \mathbb{Q}$ we thus have

$$E = \frac{1}{m_x(G_0)} \left(\sum_{j=1}^s n_j G_j - G \right). \quad \square$$

Lemma 31. *Let Y/B be an arithmetic surface such that B is local with its generic point isomorphic to the spectrum of a global field and $x \in Y$ a closed point. Let \bar{Y} be a compactification of Y over B . Then there is an open neighborhood $V \subseteq Y$ of x and a compactification \bar{X}/B of V dominating \bar{Y} such that $\bar{D} = \bar{X} - V$ is a tidy divisor and such that the following assertion holds: For every closed point $y \in \bar{X}$ and every prime divisor Z of \bar{X} passing through y there is $f \in K(\bar{X})^\times$ with support in $Z \cup \bar{D}$ such that $\deg_Z(f) > 0$ and $\deg_W(f) = 0$ for all other prime divisors W passing through y .*

Proof. Take an open neighborhood V' of x such that the complement contains all singular points except x (if x is singular) and all vertical prime divisors not passing through x and set $D' = \bar{Y} - V'$. By [9] we can replace (\bar{Y}, D') by a desingularization (in the strong sense) and thus assume that x is the only possible singular point of \bar{Y} and D' is a Cartier divisor. Choose prime divisors G_1, \dots, G_r of \bar{Y} not passing through x as in Lemma 29. Making V' smaller we may assume that G_1, \dots, G_r are contained in D' .

Let $(\bar{X}, D_0) \rightarrow (\bar{Y}, D')$ be a tidy desingularization, which exists by Proposition 3. Since \bar{Y} is regular at every point in D' , the morphism $\bar{X} \rightarrow \bar{Y}$ is a consecutive blowup in closed points over D' . Moreover, the exceptional fibre

of each blowup in a closed point z is isomorphic to $\mathbb{P}_{k(z)}^1$ (see [10], Chapter 8, Theorem 1.19). Denote by E_1, \dots, E_n the irreducible components of the exceptional locus of $\bar{X} \rightarrow \bar{Y}$. For each $i = 1, \dots, n$ choose two different closed points $y_i, z_i \in E_i$ in the regular locus of D_0 and (horizontal) prime divisors D_i and K_i intersecting E_i transversally at y_i and z_i , respectively. Since a horizontal prime divisor consists of only two points, namely the special and the generic point, D_i and K_i are regular and do not intersect D_0 in any other point. Denote by \bar{D} the sum of D_0 and the prime divisors $D_1, \dots, D_n, K_1, \dots, K_n$ as above and set $V = \bar{X} - \bar{D}$.

We claim that (\bar{X}, \bar{D}) has the required properties. By the definition of a tidy desingularization, D_0 is a tidy divisor. The property of being tidy is invariant under adding horizontal prime divisors intersecting the special fibre transversally in a regular point of D_0 . Therefore, also \bar{D} is tidy. Let $y \in \bar{X}$ be a closed point and Z a prime divisor of \bar{X} passing through y . Either G_1, \dots, G_s or G_{s+1}, \dots, G_r do not pass through the image point of y , say G_1, \dots, G_s . Furthermore, either D_1, \dots, D_n or K_1, \dots, K_n do not pass through y , say D_1, \dots, D_n . By Lemma 30 the prime divisors $G_1, \dots, G_s, D_1, \dots, D_n$ generate the first Chow group $\text{CH}^1(\bar{X}) \otimes \mathbb{Q}$. Hence, there are $m, m_1, \dots, m_n, n_1, \dots, n_s \in \mathbb{Z}$ with $m > 0$ and $f \in K(\bar{X})^\times$ such that

$$mZ = \sum_{j=1}^n m_j D_j + \sum_{j=1}^s n_j G_j + \text{div } f.$$

The prime divisors D_1, \dots, D_n and G_1, \dots, G_s do not pass through y . Therefore, $\deg_W(f) = 0$ for all prime divisors W different from Z passing through y and $\deg_Z(f) = m > 0$. Furthermore, $D_1, \dots, D_n, G_1, \dots, G_s$ are contained in \bar{D} and thus f has support in $Z \cup \bar{D}$. \square

As a direct consequence of Lemma 31 we obtain:

Corollary 32. *In the situation of Lemma 31 let $U \subseteq V$ be a neighborhood of x such that $D' = \bar{X} - U$ is a tidy divisor. Then (\bar{X}, D') has enough tame coverings.*

We are now in the position to construct an étale neighborhood of an arithmetic surface Y/B satisfying all assumptions made in Propositions 22 and 27. Note that the assumption on the fundamental group of B is automatic in the local case. We stick to the notation of section 7

Proposition 33. *Let $\pi : Y \rightarrow B$ be an arithmetic surface of local type and \bar{x} a geometric point above a closed point $x \in Y$. Let \mathfrak{c} be a full class of finite groups such that the residue characteristic of x is not contained in $\mathbb{N}(\mathfrak{c})$. Then there is an étale neighborhood U/B' of \bar{x} and a compactification $U \subseteq \bar{X}$ of $U \rightarrow B'$ such that the complement \bar{D} of U in \bar{X} is a tidy divisor with the following properties.*

- (i) *Every connected component of D has at least one horizontal component.*
- (ii) *\bar{D}_h is nonempty and intersects all irreducible components of W .*
- (iii) *(\bar{X}, \bar{D}) has enough tame coverings.*

Proof. The arithmetic surface Y/B is the base change to B of an arithmetic surface Y_0/B_0 such that B_0 is local with generic point the spectrum of a global field and B is the completion of B_0 at its closed point. Taking completion does not affect the tidiness of a divisor, nor does it disturb properties (i) and (ii). Therefore, it suffices to prove the proposition for B local with generic point the spectrum of a global field.

Let \tilde{Y} be a compactification of Y . Choose an open neighborhood V of x and a compactification \tilde{X}/B as in Lemma 31. On every irreducible component C of \tilde{X}_b take a closed point $c_C \neq x$ in the smooth locus of C and not contained in any other irreducible component of \tilde{X}_b . For each irreducible component of \tilde{X}_b remove from V a (horizontal) prime divisor intersecting C transversally at c_C . The complement \bar{D} in \tilde{X} of the resulting open neighborhood U of x is tidy by construction and thus (\tilde{X}, \bar{D}) has enough tame coverings by Corollary 32. Moreover, (\tilde{X}, \bar{D}) has properties (i) and (ii). \square

11 The main result

We collect the work of the preceding sections in order to prove the main theorem.

Theorem 34. *Let Y/B be an arithmetic surface of local type and $\bar{y} \rightarrow Y$ a geometric point. Let \mathfrak{c} be a full class of finite groups such that the residue characteristic of B is not contained in $\mathbb{N}(\mathfrak{c})$ and for all but finitely many primes $\ell \in \mathbb{N}(\mathfrak{c})$ the extension $B[\mu_\ell] \rightarrow B$ is a \mathfrak{c} -extension. Then Y has a basis of étale neighborhoods at \bar{y} which are $K(\pi, 1)$ with respect to \mathfrak{c} .*

In particular, there exist $K(\pi, 1)$ -neighborhoods with respect to any class of finite groups of the form $\mathfrak{c}(\ell_1, \dots, \ell_n)$ for prime numbers ℓ_i prime to the residue characteristic of B .

Proof. For every étale neighborhood $V \rightarrow Y$ of \bar{y} we have to construct an étale neighborhood $U \rightarrow V$ of \bar{y} which is $K(\pi, 1)$ with respect to \mathfrak{c} . Since U is again an arithmetic surface of local type over some B' over B , we may replace Y by U and B by B' . It thus suffices to show the existence of an étale neighborhood $U \rightarrow Y$ of \bar{y} which is $K(\pi, 1)$ with respect to \mathfrak{c} .

Replacing B by a finite étale \mathfrak{c} -extension we may assume $\mu_\ell \cong \mathbb{Z}/\ell\mathbb{Z}$ on B for all prime numbers $\ell \in \mathbb{N}(\mathfrak{c})$. By Proposition 33 there is an étale neighborhood $U \rightarrow Y$ of \bar{y} and a compactification $U \subseteq \tilde{X}$ of $U \rightarrow B$ such that the complement of U in \tilde{X} is a tidy divisor satisfying properties (i)-(iii) in the statement of Proposition 33.

We have to show

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{J}_{\tilde{X}, \bar{D}, \bar{x}}} H^n(U', \Lambda) = 0 \quad (4)$$

for $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ with $\ell \in \mathbb{N}(\mathfrak{c})$ prime. For $n = 1$ the cohomology group $H^n(U, \Lambda)$ parameterizes finite étale ℓ -coverings and thus equality (4) is automatically satisfied.

Let us show equality (4) for $n \geq 2$. By proposition 22 the restriction

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{J}_{\tilde{X}, \bar{D}, \bar{x}}} H^n(X', \Lambda) \rightarrow \varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{J}_{\tilde{X}, \bar{D}, \bar{x}}} H^n(U', \Lambda).$$

is surjective. Note that the assumptions made in this proposition are among the conditions (i)-(iii) of Proposition 33 and are thus satisfied.

By Lemma 23

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(X', \Lambda) = 0$$

for $n \geq 3$. Suppose now that $n = 2$. By Proposition 27 the composition

$$\varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} \ker(H^2(X', \Lambda) \rightarrow H^0(B', R^2\pi_*\Lambda)) \hookrightarrow \varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^2(X', \Lambda) \rightarrow \varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^2(U', \Lambda)$$

remains surjective. The hypotheses of Proposition 27 are satisfied because they are part of conditions (i)-(iii) of Proposition 33. We now apply the part of Lemma 23 concerning $n = 2$, which says that

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} \ker(H^2(X', \Lambda) \rightarrow H^0(B', R^2\pi_*\Lambda)) = 0,$$

and thus also

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^2(U', \Lambda)$$

vanishes. This concludes the proof. \square

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